A CONSTRUCTIVE DEMONSTRATION OF THE UNIQUENESS OF THE CHRISTOFFEL SYMBOL

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The Christoffel symbol

$$\left\{ {\lambda \atop \mu\nu} \right\} (\mathbf{g}) = \frac{1}{2}\, g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right), \tag{1}$$

is usually introduced (Spivak, 1975) as the connection which solves the metricity condition

$$\nabla_\lambda g_{\mu\nu} = 0\,. \tag{2}$$

On the other hand, whether the Christoffel symbol is the only connection which can be constructed from a symmetric second–rank tensor $g_{\mu\nu}$ remains an open question. In this note we exhibit a constructive demonstration of the uniqueness of the Christoffel symbol.

Let us start by reviewing some simple results of tensor calculus. Let $\mathcal{M}$ be an n–dimensional differentiable manifold. Several geometric objects can be introduced over conveniently defined fibered bundles based on $\mathcal{M}$. In order to classify them we adopt a taxonomic approach, *cf.* (Visconti, 1992): a *tensor* is an object which transforms like a *tensor*, etc. The previous definition makes reference to the way in which an object transforms under changes of local coordinates. Let x^μ, $\mu = 1, 2, \cdots, n$, and y^α, $\alpha = 1, 2, \cdots, n$, be local coordinates on $\mathcal{M}$. Both sets are related by $y^\alpha = y^\alpha(x^\mu)$, and the differential form of this relation is

$$dy^\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu} \right) dx^\mu\,. \tag{3}$$

Due to the intrinsic function theorem, this relation tells us that y^α are functions of x^μ, $y^\alpha = y^\alpha(x^\mu)$. In order to express x^μ as functions of y^α, $x^\mu = x^\mu(y^\alpha)$, we need to invert eq. (3). This can be achieved if

$$\det \left(\frac{\partial y^\alpha}{\partial x^\mu} \right) \neq 0\,. \tag{4}$$

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If this is the case we have an inverse matrix $(\partial x/\partial y)$ and the relation (3) can be inverted to

$$dx^\mu = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right) dy^\alpha \,. \tag{5}$$

Then $x^\mu = x^\mu(y^\alpha)$.

Now we are ready to proceed to our taxonomic classification of geometrical objects. Relation (3) provides the first example. Let us consider a set of functions $v^\mu(x)$ and $v^\alpha(y)$ which are related in the way given by (3)

$$v^\alpha(y) = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) v^\mu(x) \,. \tag{6}$$

This defines a contravariant vector. Accordingly, a set of functions $v_\mu(x)$ and $v_\alpha(y)$ which are related by

$$v_\alpha(y) = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right) v_\mu(x) \,, \tag{7}$$

defines a covariant vector. Let us now consider the functions $\phi(x) = v_\mu(x)v^\mu(x)$ and $\phi(y) = v_\alpha(y)v^\alpha(y)$. From the relations above we easily obtain

$$\phi(y) = \phi(x) \,. \tag{8}$$

This relation defines a scalar. The relations above can be extended to tensors of different rank and covariance. For example, a covariant second–rank tensor $g_{\mu\nu}(x)$ is an object which transforms as

$$g_{\alpha\beta}(y) = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right) \left(\frac{\partial x^\nu}{\partial y^\beta}\right) g_{\mu\nu}(x) \,. \tag{9}$$

Let us now consider the derivative of the scalar function (8). We obtain

$$\frac{\partial \phi}{\partial y^\alpha}(y) = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right) \frac{\partial \phi}{\partial x^\mu}(x) \,. \tag{10}$$

Therefore, the ordinary derivative of a scalar function is a vector. Next, let us consider the ordinary derivative of a covariant vector. From (7) we obtain

$$\frac{\partial v_\beta}{\partial y^\alpha}(y) = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right) \left(\frac{\partial x^\nu}{\partial y^\beta}\right) \frac{\partial v_\nu}{\partial x^\mu}(x) + \left(\frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta}\right) v_\mu(x) \,. \tag{11}$$

Therefore, this quantity is not a vector. Let us observe that covariance is broken by a term linear in v_μ. Let us therefore introduce a new derivative

$$\nabla_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\lambda{}_{\mu\nu}\, v_\lambda \,, \tag{12}$$

where a term linear in v_μ is introduced to compensate the wrong behaviour of the last term in (11). Let us now impose that this quantity be a tensor. The result is that $\mathbf{\Gamma}$ must transform according to

$$\Gamma^{\gamma}{}_{\alpha\beta}(y) = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right)\left(\frac{\partial x^\nu}{\partial y^\beta}\right)\left[\left(\frac{\partial y^\gamma}{\partial x^\lambda}\right)\Gamma^{\lambda}{}_{\mu\nu}(x) - \left(\frac{\partial^2 y^\gamma}{\partial x^\mu \partial x^\nu}\right)\right]. \tag{13}$$

It is clear that $\mathbf{\Gamma}$ is not a tensor; this is not unexpected since in order to compensate the non–tensor character of ∂v we need a non tensor object. $\nabla_\mu v_\nu$ is the covariant derivative, and $\mathbf{\Gamma}$ is the connection.

In Riemannian geometry the natural object is the metric tensor $g_{\mu\nu}$. The metric is not related to the connection. However, we may relate them through a metricity condition. The simplest metricity condition is

$$\nabla^{\Gamma}_{\lambda} g_{\mu\nu} = 0\,. \tag{14}$$

The relation (14) has the unique solution

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}(\mathbf{g}) = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right), \tag{15}$$

which is known as the Christoffel symbol.

The Christoffel symbol is the connection which solves the metricity condition (14). It happens to be constructed from a second–rank tensor. However, there is a further question we may ask: which is the more general connection we can construct in terms of a second–rank tensor? We must therefore look for a more general scheme providing a most complete answer.

Let us therefore consider the problem of how to construct a connection $\Gamma^{\lambda}{}_{\mu\nu}$ only from a second–rank tensor $g_{\mu\nu}$. For this purpose let us remind that the connection is related to a derivation and therefore has dimensions corresponding to an inverse lenght, *i.e.*

$$\dim[\mathbf{\Gamma}] = L^{-1}\,. \tag{16}$$

On the other hand, the derivation is a linear operation. Therefore, the needed connection must be a linear combination of derivatives of $g_{\mu\nu}$, namely,

$$\Gamma^{\lambda}{}_{\mu\nu}(\mathbf{g}) = G^{\lambda\sigma\tau\rho}_{\mu\nu}(\mathbf{g},\,\mathbf{g}^{-1})\,\partial_\sigma g_{\tau\rho}\,, \tag{17}$$

where $\mathbf{G}$ is a function depending only on $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, and therefore it is a tensor. The transformation rule for $\mathbf{\Gamma}$ in (17) is

$$\begin{aligned}\Gamma^{\gamma}{}_{\alpha\beta}(y) &= G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\,\mathbf{g}^{-1})\,\partial_\delta g_{\epsilon\phi}\\ &= G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\,\mathbf{g}^{-1})\,\partial_\delta\left[\left(\frac{\partial x^\mu}{\partial y^\epsilon}\right)\left(\frac{\partial x^\nu}{\partial y^\phi}\right) g_{\mu\nu}\right]\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial y^\gamma}{\partial x^\lambda}\right) \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \left(\frac{\partial y^\beta}{\partial x^\nu}\right) \Gamma^\lambda{}_{\mu\nu}(x) \\
&\quad +2\, G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\, \mathbf{g}^{-1}) \left(\frac{\partial^2 x^\mu}{\partial y^\delta \partial y^\epsilon}\right) \left(\frac{\partial x^\nu}{\partial y^\phi}\right) g_{\mu\nu} \\
&= \left(\frac{\partial y^\gamma}{\partial x^\lambda}\right) \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \left(\frac{\partial y^\beta}{\partial x^\nu}\right) \Gamma^\lambda{}_{\mu\nu}(x) \\
&\quad +2\, G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\, \mathbf{g}^{-1})\, g_{\phi\eta} \left(\frac{\partial y^\eta}{\partial x^\lambda}\right) \left(\frac{\partial^2 x^\lambda}{\partial y^\delta \partial y^\epsilon}\right) .
\end{aligned} \tag{18}
$$

Now we require that this object transform like a connection and if we compare with (13) we obtain

$$
2\, G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\, \mathbf{g}^{-1})\, g_{\phi\eta} \left(\frac{\partial y^\eta}{\partial x^\lambda}\right) \left(\frac{\partial^2 x^\lambda}{\partial y^\delta \partial y^\epsilon}\right) = \left(\frac{\partial y^\gamma}{\partial x^\lambda}\right) \left(\frac{\partial^2 x^\lambda}{\partial y^\alpha \partial y^\beta}\right) . \tag{19}
$$

Therefore

$$
\left[4\, G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\, \mathbf{g}^{-1})\, g_{\phi\eta} - \delta^\gamma_\eta \, \left(\delta^\delta_\alpha\, \delta^\epsilon_\beta + \delta^\delta_\beta\, \delta^\epsilon_\alpha\right)\right] \left(\frac{\partial y^\eta}{\partial x^\lambda}\right) \left(\frac{\partial^2 x^\lambda}{\partial y^\delta \partial y^\epsilon}\right) = 0 \, . \tag{20}
$$

What must be zero is the symmetric part of the square bracket and then we obtain

$$
2\, G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\, \mathbf{g}^{-1})\, g_{\phi\eta} + 2\, G^{\gamma\epsilon\delta\phi}_{\alpha\beta}(\mathbf{g},\, \mathbf{g}^{-1})\, g_{\phi\eta} - \delta^\gamma_\eta \, \left(\delta^\delta_\alpha\, \delta^\epsilon_\beta + \delta^\delta_\beta\, \delta^\epsilon_\alpha\right) = 0 \, . \tag{21}
$$

Considering cyclic permutations of the indices $\delta\epsilon\phi$ we arrive to a set of equations which allow to determine $\mathbf{G}$. The solution is

$$
\begin{aligned}
& G^{\gamma\delta\epsilon\phi}_{\alpha\beta}(\mathbf{g},\, \mathbf{g}^{-1}) \\
= \;& \frac{1}{4} \left[g^{\gamma\phi} \, \left(\delta^\delta_\alpha\, \delta^\epsilon_\beta + \delta^\delta_\beta\, \delta^\epsilon_\alpha\right) + g^{\gamma\delta} \, \left(\delta^\epsilon_\alpha\, \delta^\phi_\beta + \delta^\phi_\beta\, \delta^\epsilon_\alpha\right) - g^{\gamma\epsilon} \, \left(\delta^\delta_\alpha\, \delta^\phi_\beta + \delta^\phi_\beta\, \delta^\delta_\alpha\right)\right] .
\end{aligned} \tag{22}
$$

If we now replace this result in (16) we obtain that the connection is precisely the Christoffel symbol, namely eq. (15).

One can inmediately check that

$$
\nabla^g_\lambda g_{\mu\nu} \equiv 0 \, , \tag{23}
$$

where ∇^g is the covariant derivative constructed with the Christoffel symbol.

We have shown that the Christoffel symbol is the only connection which can be constructed from a symmetric second–rank tensor. The metricity condition (23) appears just as a side result and not as the starting point for the construction of the Christoffel symbol.

References

1. M. Spivak, *A comprehensive introduction to differential geometry* (Publish or Perish, Boston, 1975).

2. A. Visconti, *Introductory differential geometry for physicists* (World Scientific, River Edge, 1992).